BIBLIOGRAPHY OF HYPER-SPACE AND NON-EUCLIDEAN GEOMETRY.

By George Bruce Halsted, *Tutor in Princeton College, N. J.*

Until the present century the Euclidean Geometry was supposed to be the only possible form of Space-science; that is, the space analysed in Euclid's axioms was supposed to be the only non-contradictory sort of space.

The Parallel-Postulate was generally supposed to be a consequence of the nature of straight lines, and demonstrable from the remaining postulates and axioms. The researches of Peyrard show that it was not given out by Euclid as an axiom, since in all the MSS. examined by him it is kept separate from the axioms, and has only been classed with them by an obvious error in modern times. The enormous number of unsatisfactory attempts to prove this postulate, led finally to a systematic development of the results obtainable when it is denied, and then sprang forth the Non-Euclidean or Absolute Geometry.

In the "Encyclopædie der Wissenschaften und Künste; von Ersch und Gruber; Leipzig, 1838; under "Parallel," Sohncke says that in mathematics there is nothing over which so much has been spoken, written and striven, as over the theory of parallels, and all, so far, (up to his time) without reaching a definite result and decision. He divides the attempts into three classes:—1. In which is taken a new definition of parallels. 2. In which is taken a new axiom different from Euclid's. But just as Euclid's cannot be considered axiomatic, so is it with these new postulates. This led to the third, the largest and most desperate class of attempts, namely, to deduce the theory of parallels from reasonings about the nature of the straight line and plane angle. The article is followed by a carefully prepared list of ninety-two authors on the subject, from the earliest times up to the year 1837.

In English an account of like attempts is given in the "Geometry without Axioms," by Perronet Thompson: Cambridge, 1833; where the author also makes an elaborate attempt of his own. These accounts may be considered to bring the subject up to the point where, through the perfectly original

works of two new geometers, it assumed a totally new aspect and became the question of Non-Euclidean Geometry. At this point we take up its Bibliography, together with that of Hyper-Space, which, though springing at first from a purely analytical basis, has become intimately connected with the former.

1. LOBATCHEWSKY, NICOLAUS IVANOVITCH. (1793–1856).

The first public expression of his discoveries was given in a discourse at Kasan, February 12, 1826.

I. Principien der Geometrie. Kasan, 1829–30.

II. Neue Anfangsgruende der Geometrie, mit einer vollständigen Theorie der Parallelen. Gelehrte Schriften der Universität Kasan, 1836–38. His chief work (orig. pub. in Russian). Hoüel has made a translation of it into French (in MS.)

III. Geometrie Imaginaire. Crelle's Journal, B. XVII, pp. 295–320. 1837.

IV. Application de la Geometrie Imaginaire à quelques Integrales. Crelle. 1836.

V. Geometrische Untersuchungen zur Theorie der Parallellinien. Berlin, 1840. 61 pages.

VI. Pangeometrie, ou precis de geometrie fondée sur une theorie generale et rigoureuse des paralleles. Imprimerie de l'Université. Kazan, 1855. This, originally published in French, has been translated into Italian by G. Battaglini: Giornale di Matematiche. Anno V, Settembre e Ottobre, 1867, pp. 273–320. It is also given by Erman, Archiv Russ. XVII, 1858, pp. 397–456. V has also been translated and published in French; see Hoüel.

2. GAUSS, C. J.

I. Briefwechsel zwischen Gauss und Schumacher. See especially the letters of 17 May and 12 July, 1831. Bd. 2, pp. 268–271.

"La Géometrie non-Euclidienne ne renferme en elle rien de contradictoire, quoique, à première vue, beaucoup de ses resultats aient l'air de paradoxes. Ces contradictions apparents doivent être regardées comme l'effet d'une illusion, due à l'habitude que nous avons prise de bonne heure de considérer la géométrie Euclidienne comme rigoureuse."

II. Werke. Bd. IV, p. 215. This reference is to the researches presented in 1827 to the Society of Göttingen under the title: "Disquisitiones generales circa superficies curvas."

3. BOLYAI, WOLFGANG AND JOHANN.

I. Tentamen Juventutem studiosam in elementa Matheseos purae, elementaris ac sublimioris, methodo intuitiva, evidentique huic propria, introducendi. Tomus Primus, 1832 Secundus, 1833. 8o. Maros-Vásárhelyini. These two volumes, published by subscription, form the principal work of Wolfgang Bolyai. In the first volume, with special title page and numbering, appeared the celebrated Appendix of Johann Bolyai,

II. Ap., scientiam spatii *absolute veram* exhibens: a veritate aut falsitate Axiomatis XI Euclidei (a priori haud unquam decidenda) independentem. Auctore Johanne Bolyai de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Captaneo. Maros-Vásárhely., 1832. (26 pages of text). This celebrated Appendix has been translated into French, see Hoüel, into Italian, see Battaglini, and into German, see Frischauf.

III. The last work of Wolfgang Bolyai, the only one he composed in German, is entitled: Kurzer Grundriss eines Versuches, I. die Arithmetik, durch zweckmässig construirte Begriffe, von eingebildeten und unendlichkleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen: II. In der Geometrie, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krummen, der verschiedenen Arten der Gleichheit u. dgl. nicht nur scharf zu bestimmen, sondern auch ihr Sein in Raume zu beweisen: und da die Frage, *ob zwei von der dritten geschnittene Geraden, wenn die Summa der inneren Winkel nicht* $= 2R$, *sich schneiden oder nicht?*, niemand auf der Erde ohne ein Axiom (wie Euclid das XI) aufzustellen, beantworten wird; die davon unabhängige Geometrie abzusondern, und eine auf die Ja Antwort, andere auf das Nein so zu bauen, dass die Formeln der letzen auf ein Wink auch in der ersten gültig seien. Maros-Vásárhely., 1851. 8o. (88 pages of text). The author mentions Lobatchewsky's Geometrische Untersuchungen, Berlin, 1840, and compares it with the work of his son Johann Bolyai, "au sujet duquel il dit: 'Quelques exemplaires de l'ouvrage publié ici ont été envoyés à cette époque à Vienne, à Berlin, à Göttingen. . . . De Goettingen, le géant mathématique, [Gauss]

qui du sommet des hauteurs embrasse du même regard les astres et la profondeur des abîmes, a écrit qu'il était ravi de voir exécuté le travail qu'il avait commencé pour le laisser après lui dans ses papiers.' "

4. JACOBI, C. G. J.

I. De binis quibuslibet functionibus homogeneis, &c. Crelle Journ. XII, 1834. 1–69.

Several papers in the early volumes of Crelle *in effect* relate to the transformation of coordinates, and the attraction of spherical shells, &c., in n-dimensional space, but the treatment is throughout analytical and there is no especial reference to space of four or more dimensions.

5. GRASSMANN, H.

I. Die lineale Ausdehnungslehre. Leipzig, 1844. 2d Ed., 1878.

II. Die Ausdehnungslehre. Berlin, 1862.

Grassmann was perhaps the first who developed the theory of extended manifoldness, as a special case of which appears the theory of space. But his manifoldness differs from our space only as being a generalisation of it by increasing the number of dimensions while preserving relative properties of position and measure. In a word it is homaloidal Hyper-space, and does not open so wide and diverse a field as Riemann's profound paper.

6. CAYLEY, ARTHUR.

I. Chapters in the Analytical Geometry of (n) Dimensions. Camb. Math. Journ., IV, 1845. pp. 119–127.

II. Sixth Memoir upon Quantics. Phil. Trans., vol. 149.

III. On the Non-Euclidean Geometry. Clebsch, Ann. V, 630–634. 1872.

IV. A Memoir on Abstract Geometry. Phil. Trans., CLX, 51–63. 1870.

V. On the superlines of a quadric surface in five dimensional space. Quarterly Journ., vol. XII, 176–180. 1871–2.

In his Memoir on the principles of an Abstract m-dimensional Geometry (IV), Prof. Cayley says: "The science presents itself in two ways,—as a legitimate extension of the ordinary *two-* and *three-*dimensional geometries; and as a need in these geometries and in analysis generally. In fact whenever we are concerned with quantities connected together in any manner, and

which are, or are considered as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as the coordinates of a point in a plane or in space: for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation; but this can only be obtained by means of the notion of a space of the proper dimensionality; and to use such representation, we require the geometry of such space. An important instance in plane geometry has actually presented itself in the question of the determination of the number of the curves which satisfy given conditions: the conditions imply relations between the coefficients in the equation of the curve; and for the better understanding of these relations it was expedient to consider the coefficients as the coordinates of a point in a space of the proper dimensionality."

7. SYLVESTER, J. J.

I. On certain general Properties of Homogeneous Functions. Cam. and Dub. M. Journ., Feb. 1851.

II. Partitions of Numbers. (Lectures). London, 1859.

III. Barycentric Projections. Phil. Mag., or Br. Assoc'n.

IV. Inaugural Address to Math. Section, British Association at Exeter, August, 1869. Nature, vol. I, p. 238. Republished with Notes in "Laws of Verse." Longmans. 1870.

8. RIEMANN, B.

I. Ueber die Hypothesen welche der Geometrie zu Grunde liegen. Habilitationsschrift von 10 Juni, 1854. Abhandl. der König. Gesellsch. zu Göttingen. B. XIII. Reprinted in "B. Riemann's G. M. Werke," Leipzig, 1876.

This profound paper is difficult reading. Frischauf has attempted to make the study of it easier by giving, in his Absolute Geometry, notes and references at points where Riemann has given results while suppressing processes.

It has been translated into French by Hoüel. Annali di Mat., serie II, tome III, fasc. IV, 309–327. 1870.

The position of a point in space being determined by three quantities, x_1, x_2, x_3, to a continuous change of that position corresponds a continuous

variation of these three quantities. Then Riemann holds that the measure of the distance between the point $(x_1 x_2 x_3)$ and the next point $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ is not necessarily the square root of the sum of the squares of the three differentials.

If the sides of a triangle constructed on a given sphere be all of them increased or diminished in the same proportion, the shape of the triangle will not remain the same. On the contrary, the figures constructed in a plane may be magnified or diminished to any extent without alteration of shape.

Riemann found that this property of the plane is equivalent to the two following axioms: (1) That two geodesic lines which diverge from a point will never intersect again, or, as Euclid puts it, that two straight lines cannot enclose a space; and (2) that two geodesic lines which do not intersect will make equal angles with every other geodesic line. Deny the first of these axioms, and you have a manifoldness of positive curvature; deny the second, and you have one of negative curvature. The plane lies midway between the two, and its curvature is zero at every point. Thus Riemann found three different sorts of geometry. Bolyai had only noticed two. Also here for the first time was brought forward the distinction between "unbegrenzte" and "unendliche" "Unendliche" is our "infinite." A series is "unbegrenzte" when, without inversion of the derivation process, one can go on continually. If one by continued forward application of this process comes back to the starting point, the series is finite; but if the process can go on continually without ever coming again to any previous term, the series is infinite. The like parts of a circle may serve as an example of a series which though finite is yet unbegrenzt, for we may pass continually on from one to the next forever. Now we rightly attribute to space this property of being without a bound, for a limit to it is contradicted by its homogeneousness. But from this it in no way follows that space is infinite.

9. SALMON, GEORGE.

I. Lessons on Modern Higher Algebra. 1866. p. 212, &c.

II. Extension of Chasles' Theory of Characteristics to Surfaces.

10. BALTZER, R.

I. Elements of Mathematics. Dresden, 1866.

II. Ueber die Hypothesen der Parallelentheorie. C. Journal. Band 83, s. 372. Berichte der K. s. G. zu Leipzig. T. XX, 95–96. 1868.

11. HOÜEL, J.

I. Études Géométriques sur la Theorie des Parallels, par Lobatchewsky; suivi d'un extrait de la correspondance de Gauss et de Schumacher. Paris, 1866. 8o.

II. Essai critique sur les Principes fondamentaux de la Geometrie. Paris, 1867. 8o.

III. La Science Absolue de l'Espace independante de la vérité ou de la fausseté de l'Axiome XI d'Euclide (que l'on ne pourra jamais établir *a priori*); par Jean Bolyai: précédé d'une Notice sur la Vie et les Travaux de W. et J. Bolyai, par M. Fr. Schmidt. Paris, 1868. 8o.

IV. Sur les hypotheses qui servent de fondement a la geometrie, memoire posthume de B. Riemann. Annali di Mat, serie II, tome III, fasc. IV, 309–327. 1870.

V and VI. Beltrami's "Geometria non-Euclidea" and "Spazii de curvatura costante," translated into French. Annales Scien. de l'Ecole Normale Supérieure, tome VI. 1869.

VII. Note sur l'impossibilité de demontrer par une construction plane le principe de la theorie des paralleles dit Postulatum d'Euclide. Memoires de la Société des Sciences de Bordeaux, tome VIII. 1870–72. Paris: J. B. Ballière.

VIII. Du rôle de l'experience dans les sciences exactes. Prague, 1875. Translated into German by F. Müller. Grunert's Archiv., vol. 59, p. 65.

12. BELTRAMI, E.

I. Risoluzione del problema di riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette. Annali di Mat., tome VII. 1866.

II. Saggio di Interpretazione della Geometria non-Euclidea. Naples, 1868. Giornale de Matematiche, (G. Battaglini,) Anno VI, pp. 284-312.

III. Teoria fondamentale degli Spazii di Curvatura costante. Annali di Mat., ser. II, tome II. Milano, 1868.

By the curvature of a system Riemann and Beltrami understand the relation of the area of an infinitesimal triangle of the system to the corresponding area of a system of constant positive curvature (système sphérique). It is in this sense that Beltrami's pseudospherical systems have a constant negative curvature.

This differs from what Kronecker calls the curvature of a system or, at the end of his Memoir, the condensation of the system, which instead of the relation of the areas of infinitesimal triangles, means the relation of the volumes of infinitesimal tetrahedrons.

IV. Theoreme de Geometrie pseudospherique. Giornale di Mat. This shows the connection between certain straight lines in the non-Euclidean plane and the curve whose tangents are of a constant length in the Euclidean plane.

V. Sur la surface de revolution qui sert de type aux surfaces pseudospheriques. Giornale di Mat. (G. Battaglini), tome X. 1872. This contains several theorems relative to the surface of revolution having for meridian the curve whose tangents are of a constant length.

13. Battaglini, G.

I. Sulla Geometria Immaginaria di Lobatchewsky. Giornale di Mat., Anno V, pp. 217–231. 1867. In this the author reaches by a different method most of Lobatchewsky's results.

II. Pangeometria o sunto di geometria fondata sopra una teoria generale e rigorosa delle parallele, per N. Lobatchewsky, (versione del Francese). Giornale di Mat., Anno V, pp. 273–320.

III. Sulla scienza dello spazio assolutamente vera, ed independente dalla verita o dalla falsita dell' assioma XI di Euclide: per Giovanni Bolyai, (versione dal latino). Giornale di Mat., Anno VI, pp. 97–115. 1868.

14. Helmholtz, H.

I. Ueber die Thatsachen die der Geometrie zum Grunde liegen. Nachrichten, Göttingen, Juni 3, 1868.

II. Sur les faits qui servent de base à la Geometrie. Memoires de la Soc. des Sciences de Bordeaux. 1868.

III. The Origin and meaning of Geometrical Axioms. Part I, Mind, No. III. July, 1876. Some of this article had been previously given in the Academy, Feb. 12, 1870, vol. I, p. 128. Replied to by Jevons; Nature, vol. IV, p. 481. Jevons' ideas developed by J. L. Tupper; Nature, vol. V, p. 202. Replied to by Helmholtz; Academy, vol. III, p. 52. Part II, Mind. April, 1878.

15. POTOCKI, S.

Notice historique sur la vie et les travaux de N. I. Lobatchewsky. Bulletino di Bibliographia du Prince Boncompagni, tome II, 223. May, 1869. Translated from the discourse of Janichefsky, who was editing a new edition of Lobatchewsky's works.

16. DARBOUX, G.

I. Comptes Rendus de l' Acad. Aug., 1869.

II. Sur les equations aux dérivées partielles du second ordre. Comp. R. LXX. 1870. I, 673; II, 746.

17. KRONECKER, L.

I. Ueber Systeme von Functionen mehrer Variabeln. Monatsbericht der Kgl. Akademie zu Berlin. Part I, März, 1869. Part II, August, 1869. The generalized spaces treated here are mostly supposed homaloidal. The author mentions the power given him by considerations of geometry of position in overcoming algebraical difficulties.

18. CHRISTOFFEL, E. B.

I. Allgem. Theorie d. geodät. Dreiecke. Berlin, 1869.

II. Ueber die Transformation der homogenen Differentialausdrücke 2[ten] Grades. Borchardt's Journal, LXX, 46–70. 1870.

III. Ueber ein betreffendes Theorem. Borchardt's Journal, LXX, 241-245. 1870. II and III treat of n dimensions.

19. CLIFFORD, W. K.

I. On Probability. Educational Times.

II. Lecture on "the Postulates of the Science of Space." The extent of space may be a finite number of cubic miles. He says, "In fact, I do not mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things."

III. Preliminary sketch of Biquaternions. Proceedings of L. Math. Soc., IV, 381–395. The author shows that the symbols have a more general interpretation in the geometry of three dimensions which Klein calls the elliptic in distinction from the parabolic or Euclidean geometry.

20. LIPSCHITZ, R.

I. Untersuchungen in Betreff die ganzen homogenen Functionen von n Differentialen. Borchardt's Journal, Bde. LXX, 3, pp. 71–102. LXXII, 3, pp. 1–56. Analysed in the proceedings of the Berlin Academy, Jan., 1869, pp. 44–53. An analysis by the author is given in the Bulletin des Sciences Mathematiques, tome IV; I, pp. 97–110; II, pp. 142–157. Paris, 1873. The author demonstrates that the general form of the linear element of a system of three dimensions can be referred back to the form given by Riemann for a system of constant curvature, when a certain condition necessary and sufficient, (that the measure of the constant curvature shall be equal to a given function) is satisfied.

II. Entwickelung einiger Eigenschaften der quadratischen Formen von n Differentialen. Borchardt's Journal, LXXI, 274–287, 288–295. Bulletin des Sciences Math., IV, 297-307; V, 308–314. Paris, 1873.

III. Untersuchung eines Problems der Variationsrechnung. Borchardt's Journal, Bd. LXXIV, pp. 116–149, 150–171. Bulletin, tome IV, 212–224, 297–320.

IV. Extension of the Planet-problem to a space of n dimensions and of constant integral curvature. Translated by A. Cayley. Quar. Jour. Math. XII, 349–370. 1871.

21. GENOCCHI, A.

Dei primi principii della mecanica e della geometria in relazione al postulato d'Euclide. Firenze, 1869. Accademia da XL in Modena, serie III, tomo II, parte I. This memoir connects the theory of parallels and parallel forces with mechanical laws and considerations.

22. NÖTHER, M.

Zur Theorie der algebraischen Functionen mehrerer complexer Variabeln. Göttingen, Nachrichten. 1869.

23. BETTI, E.

Sopra gli spazi di un numero qualunque di dimensioni. Annali di Mat., 2 série, IV, pp. 140–158. 1870. Contains analytical treatment of the properties and relations of spaces of equal or different dimensions.

24. de Tilly, M.

I. Études de mechanique abstraite. Memoires couronnés de l'Academie royale Belgique, tome XXI.

II. Report on a letter from Genocchi to Quetelet. Bulletin de Belg. (2) XXXVI, 124–139.

25. Becker, J. K.

I. Abhandlungen aus dem Grenzgebiete der Mathematik und Philosophie. Zürich, 1870. (62 pages).

II. Ueber die neuesten Untersuchungen in Betreff unserer Anschauungen vom Raume. Schlömilch Zeitschrift, XVII, 314–332. 1872. For recension of I see XV, 93.

III. Die Elemente der Geometrie auf neuer Grundlage. Berlin, 1877. (300 pages). This contains a systematic statement of the ground-principles of the plane and of space which appear in the properties of the simplest figures. The moving idea is that all the properties of figures are grounded in the nature of space itself.

26. Schlaefli, L.

I. Nota alla memoria del Sig. Beltrami sugli spazie dela curvatura costante. Annali di Mat, 2d serie, t. V, 178–193. 1870.

II. Beltrami. Osservazione sulla precedente Memoria del Sig. Prof. Schläfli. Brioschi, Ann. V, 194–198. A theorem of Beltrami leads to the problem: To distinguish all spaces of n-dimensions in which any geodetic line is represented by a system of $n - 1$ linear equations. Schläfli shows that only spaces of constant curvature fulfil this condition.

27. Beez, R.

I. Ueber conforme Abbildung von Mannigfaltigkeiten höherer Ordnung. Schlömilch Zeits., XX, 253–270.

II. Zur Theorie des Krummungsmasses von Mannigfaltigkeiten höherer Ordnung. Schlömilch Z., XX, 423–444. Fortsetzung, XXI, 373–401. The author shows that Kronecker's generalized expression for the measure of curvature for Hyper-space cannot, as in tridimensional space, be represented by the coefficients of the expression for the linear element. In reference to this, Lipschitz (Beitrag zur Theorie der Krummung. Borchardt's Journ.,

LXXXI, 239, Note) remarks, that to make the representation possible, one has only to take in addition "die Differentialquotienten jener Coefficienten nach den Variabeln."

28. ROSANES, J.

Ueber die neuesten Untersuchungen in Betreff unser Anschauung vom Raume. Breslau, 1871. 8o. This is an elementary exposition of the ideas contained in Riemann's celebrated paper.

29. FLYE, ST. MARIE.

I. Sur le postulatum d'Euclide. L'Institut, I, sect. XXXVIII, 53–54. 1870. That the postulatum of Euclid cannot be proved except by assuming another of like value.

II. Études analytiques sur la théorie des paralleles. Paris, 1871. 8o. Treats of a system of coordinates whose axis of x is a circle with infinite radius.

30. LIE, SOPHUS.

I. Ueber diejenige Theorie eines Raumes mit beliebig vielen Dimensionen, die der Krümmungs-Theorie des gewöhnlichen Raumes entspricht. Göttingen, Nachrichten. May, 1871. In this many geometrical theorems are extended to a space of any number of dimensions.

II. Zur Theorie eines Raumes von n Dimensionen. Göttingen, Nachrichten. Nov., 1871. 535–557. The sphere of n dimensions is used as element of a space of $n + 1$ dimensions.

31. KLEIN, FELIX.

I. Ueber die sogenannte Nicht-Euklidische Geometrie. Göttingen, Nachrichten. August, 1871. Math. Ann., IV, 573–625; VI, 112–145. The projective geometry is proved to be independent of the theorem of parallels. See Jahrbuch über die Fortschritte der Math., 1873.

II. Ueber neuere geometrische Forschungen. Erlangen, 1872.

32. SALETA, F.

Exposé sommaire de l'idée d'espace au point de vue positif. Paris, 1872. (32 pages.) The author considers the axioms and postulates of geometry as definitions of the kind of space treated.

33. KÖNIG, J.

Ueber eine reale Abbildung der Nicht-Euklidischen Geometrie. Göttingen, Nachrichten. March, 1872. (7 pages). A study of the relations which exist between the non-Euclidean geometry and the geometry of complexes.

34. JORDAN, CAMILLE.

I. Essai sur la Geometrie à n Dimensions. Comptes Rendus, LXXV, 1614–1617. 1872. Bulletin de la Soc. Math., tome III, pp. 104, &c., IV, p. 92.

II. Sur la theorie des courbes dans l'espace à n dimensions. Comptes Rendus, LXXIX, p. 795. 1874.

III. Généralisation du théorème d'Euler sur la courbure des surfaces dans l'espace à $m + k$ dimensions. Compt. Rendus, LXXIX, p. 909.

35. FRISCHAUF, J.

I. Absolute Geometrie, nach J. Bolyai. Leipzig, 1872. 8o. XII–96 pp.

II. Elemente der Absoluten Geometrie. Leipzig, 1876. VI–142 pages.

36. KOBER, J.

On infinity and the new geometry. Zeits. fur Math. Unterricht. 1872.

37. HOFFMANN, J. C. V.

Resultate der Nicht-Euklidischen oder Pangeometrie. Zeits. für Math. Unterricht, IV, 416–417.

38. FREYE, G.

Ueber ein geometrische Darstellung der imaginairen Gebilde in der Ebene. Jena. Neuenhahm.

39. CASSANI, P.

Intorno alle ipotesi fondamentali della geometria. Battaglini, G. XI, 333–349.

40. FRAHM, W.

Habilitationsschrift. Tübingen, 1873.

41. LINDEMANN, F.

Ueber unendlich kleine Bewegungen starrer Körper bei allgemeiner projectivischer Massbestimmung. Erlang., Ber., 1873, 28 Juli. Clebsch, Ann. VII, 56-144.

42. D'OVIDIO, E.

Studio sulla geometria projettiva. Brioschi, Ann. (2) VI, 72–101.

43. STAHL, H.

Ueber die Massfunctionen der analytischen Geometrie. Berlin, 1873.

44. SCHERING, E.

Linien, Flächen und höhere Gebilde im mehrfach ausgedehnten Gauss' schen und Riemann' schen Raume. Göttingen, Nachrichten, 13–21; 149–159. 1873. For a notice of the last seven by Klein, see Jahrbuch über die Fortschritte der Math. Berlin, 1875.

45. SPITZ, C.

Die ersten Sätze vom Dreiecke und den Parallelen. Nach Bolyai's Grundsätzer. Leipzig. For notice see Grunert's Archiv., LVII, Litber, CCXXVI, 10.

46. HALPHEN, G.

Recherches de géométrie à n dimensions. Bull. Soc. Math., F. II, 34-52. On the projective properties of structures in Hyper-space.

47. ESCHERICH.

Die Geometrie auf den Flachen constanter negativer Krummung. Kais. Akad. Bd. LXIX.

48. SPOTTISWOODE, W.

I. Sur la representation des figures de géométrie à n dimensions par les figures correlatives de géométrie ordinaire. Comptes Rendus, LXXI, 875–877.

II. Nouveaux exemples de la representation, par des figures de géométrie, des conceptions analytiques de géométrie à n dimensions. C. R., LXXXI, 961–963. Geometrical interpretation of analytical space of more than three dimensions.

49. LEWES, G. H.

Imaginary Geometry and the truth of Axioms. Problems of Life and Mind, 1st series, vol. II. London, 1875.

50. FUNCKE.

Grundlagen der Raumwissenschaft. Hannover, 1875. Proposes a case in which he holds it would be imperatively necessary to suppose a fourth dimension.

51. Zöllner, J. C. F.

I. Principien einer Elektrodynamischen Theorie der Materie. Leipzig, 1876. See Review by Prof. Carl Stumpf, Phil. Monatshefte, B. XIV, 13–30.

II. Wissenschaftliche Abhandlungen. Leipzig, 1878. See Review by P. G. Tait, Nature, March 28, 1878, pp. 420–422.

52. Frank, A. von.

Der Körperinhalt des senkrechten Cylinders und Kegels in der absoluten Geometrie. Grunert's Archiv., vol. 59, p. 76.

53. Günther, Siegmund.

Ziele und Resultate der neuern Math. Histor. Forschung. Erlangen, 1876. Also Reviews, Grunert's Archiv., Theil 60.

54. Réthy.

Die Fundamental gleichungen der nicht-Euklidischen Trigonometrie auf elementarem Wege abgeleitet. Grunert's Archiv., LVIII, 416.

55. Frankland, W.

On the simplest continuous manifoldness of two dimensions and of finite extent. Nature, vol. 15, No. 389, April 12, 1877, pp. 515–517. This article called forth an objection from C. J. Monroe in Nature, vol. 15, No. 391, April 26, 1877, p. 547, where he claims that it necessitates that a perpendicular change sign without passing through Infinity or vanishing. Of this objection Prof. Newcomb says: "I cannot see even what it consists in. The first elements of complex functions imply that a line can change direction without passing through Infinity or zero."

56. Erdmann, Benno.

Die Axiome der Geometrie. (Untersuchung der Riemann-Helmholtz Raum theorie). Leipzig, 1877.

57. Mehler.

Ueber die Benutzung einer vierfachen Manningfaltigkeit zur Ableitung orthogonaler Flachensysteme. Borchardt's Journ., Band 84, pp. 219–230. December, 1877.

58. CANTOR, G.

Ein Beitrag zur Mannigfaltigkeitslehre. Borchardt's Journ., Band 84, pp. 242–258. December, 1877.

59. NEWCOMB, SIMON.

I. Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature in the fourth dimension. Borchardt's Journ., Band 83, pp. 293–299. 1877.

This article, founded on the ideas of Riemann, considers the subject from the standpoint of elementary geometry. "It may also be remarked that there is nothing within our experience which will justify a denial of the possibility that the space in which we find ourselves may be curved in the manner here supposed."

II. Note on a class of Transformations which Surfaces may undergo in Space of more than three dimensions. American Journal of Math., I, pp. 1–4. 1878.

"If a fourth dimension were added to space, a closed material surface (or shell) could be turned inside out by simple flexure; without either stretching or tearing."

60. TANNERY, PAUL.

I. La Geometrie Imaginaire et la Notion d'Espace. Revue Philosophique, Nov., 1876, 433–451. II, No. 6, pp. 553-575, Juin, 1877.

61. LÜROTH, J.

Ueber Bertrand's Beweis des Parallelenaxioms. Schlömilch Zeitschrift, XXI, pp. 294–297. 1876. Pointing out the failure of that attempted demonstration.

62. WEISSENBORN, H.

Ueber die neueren Ansichten vom Raum und von den geometrischen Axiomen. Vierteljahrschrift für Wissenschaftliche Philosophie, II, Zweites Heft, Erstes Artikel, 222–239. 1878.